A Constructive Proof That There Are Infinitely Many Primes

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Introduction

We describe an algorithm that, given any positive integer n, produces n distinct prime numbers.

Algorithm

1. **Initialize:** Let S_0 be any finite set of distinct prime numbers. (e.g., $S_0 = \{2, 7, 11\}$).
2. **Generate a New Candidate Number:** Define
$$x_i = \left(\prod_{s \in S_i} s \right) + 1$$
(e.g., $x_0 = 2 \times 7 \times 11 + 1 = 155$).
3. **Find New Primes:** Let P_i be the set of all prime factors of x_i. (e.g., $155 = 5 \times 31$ so $P_0 = \{5, 31\}$).
4. **New Primes Are Disjoint from S_i:**
   - Since $x_i \equiv 1 \mod s$ for all $s \in S_i$, none of the primes in S_i divide x_i.
   - Therefore, every prime in P_i is a new prime not found in S_i.
5. **Update the Set:**
$$S_{i+1} = S_i \cup P_i$$
(e.g., $S_1 = \{2, 7, 11, 5, 31\}$).
6. **Repeat Until $|S_i| \geq n$:**
   - Once $|S_i|$ contains at least n elements, return n primes from S_i.

Since each iteration introduces at least one new prime, this process always terminates for any n. The existence of this algorithm proves that there are infinitely many primes.

Example

We illustrate the algorithm for 5 iterations, starting with $S_0 = \{2, 7, 11\}$.

i	S_i	x_i	P_i
0	$\{2, 7, 11\}$	155	$\{5, 31\}$
1	$\{2, 7, 11, 5, 31\}$	2407	$\{17, 19, 73\}$
2	$\{2, 7, 11, 5, 31, 17, 19, 73\}$	487969	$\{13, 37, 97\}$
3	$\{2, 7, 11, 5, 31, 17, 19, 73, 13, 37, 97\}$	267711443	$\{3, 421, 2113\}$
4	$\{2, 7, 11, 5, 31, 17, 19, 73, 13, 37, 97, 3, 421, 2113\}$	1123034197009	$\{7, 149, 123863\}$

Implementation

Below is a Python implementation of the algorithm:

```
import sympy

def generate_primes(n):
    S = {2, 7, 11}
    while len(S) < n:
        x = sympy.prod(S) + 1
        new_primes = set(sympy.factorint(x).keys())
        S.update(new_primes)
    return sorted(S)[:n]

# Example usage
generate_primes(20)
# output: [2, 3, 5, 7, 11, 13, 31, 73, 109,
  421, 577, 8059, 30631, 76471, 245209,
  987523, 243941329, 526827139, 2228092512841944575931,
  785471215267242156893680003]
```